

A cosmological model in Weyl-Cartan spacetime:

I. Field equations and solutions

Dirk Puetzfeld

Institute for Theoretical Physics, University of Cologne, 50923 Köln, Germany

E-mail: dp@thp.uni-koeln.de

Abstract. In this first article of a series on alternative cosmological models we present an extended version of a cosmological model in Weyl-Cartan spacetime. The new model can be viewed as a generalization of a model developed earlier jointly with Tresguerres. Within this model the non-Riemannian quantities, i.e. torsion T^α and nonmetricity $Q_{\alpha\beta}$, are proportional to the Weyl 1-form. The hypermomentum $\Delta_{\alpha\beta}$ depends on our ansatz for the nonmetricity and vice versa. We derive the explicit form of the field equations for different cases and provide solutions for a broad class of parameters. We demonstrate that it is possible to construct models in which the non-Riemannian quantities die out with time. We show how our model fits into the more general framework of metric-affine gravity (MAG).

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1. Introduction

At the moment cosmology is one of the fastest changing fields in physics. This fact might, on the one hand, be ascribed to the vast amount of new observational data (cf [15, 16, 17] e.g.), on the other hand there are still fundamental open questions within what is nowadays called the cosmological standard model [1, 2]: Where does the inflaton field come from? Is there something like the cosmological constant λ which contributes to the dark energy etc.?

From a theoretical viewpoint one might divide efforts today within cosmology into two broad subclasses. Firstly, we have models which extend the standard model to a certain amount, inflation [18, 19], e.g., can be viewed as an add-on for the classical Friedman-Lemaître-Robertson-Walker (FLRW) model. All of these models have in common that they do not affect the structure of spacetime itself, i.e., they are still bound to a four-dimensional Riemannian spacetime and, in addition, do not modify the underlying gravity theory, i.e., General Relativity (GR). Secondly, we have models which are no longer tied to a four-dimensional Riemannian spacetime and might also modify the underlying gravity theory. One example for the latter is the so-called metric-affine gauge theory of gravity (MAG), as proposed by Hehl *et al* in [5]. Within this gauge theoretical formulation of a gravity theory there are new geometrical quantities, torsion $T^\alpha := D\vartheta^\alpha$ and nonmetricity $Q_{\alpha\beta} := -Dg_{\alpha\beta}$, which liberate spacetime from its Riemannian structure. In this paper we are going to work within this more general framework of a post-Riemannian gravity theory.

The main reason to consider more general structures within cosmology is the idea that new geometrical quantities might shed light on the problems of the cosmological standard model, e.g. provide an explanation for the rather artificial introduction of an additional scalar field, like the inflaton field. The new quantities couple to the spin, shear, and dilation currents of matter, which are supposed to come into play at high energy densities, i.e. at early stages of the universe [9, 10]. Another open question, which might also be attacked, is the origin of the large amount of dark energy, predicted by recent supernova observations. A model which deals with a new kind of dark matter interaction within a post-Riemannian theory has been proposed by Tucker and Wang in [12].

Within this paper, we confine ourselves to a Weyl-Cartan spacetime. This type of spacetime can be viewed as a special case of the more general metric-affine framework, in which the tracefree part of the nonmetricity $Q^\alpha_{\alpha\beta}$ vanishes by definition. The reason to consider this kind of restriction is twofold. Firstly, computations are more feasible in a spacetime which is not endowed with the full MAG symmetries. Secondly, the Weyl-Cartan spacetime is, unlike the Riemann-Cartan or the Weyl spacetime, still able to carry both of the new field strengths nonmetricity and torsion. Let us note that the metric-affine framework incorporates all of the above mentioned types of spacetimes. The Einstein-Cartan theory, which is formulated in a Riemann-Cartan spacetime, represents a viable gravity theory with torsion. By switching off all non-

Riemannian quantities in MAG we arrive at GR.

As becomes clear from the title, this paper stands at the beginning of series of articles on cosmological models in alternative gravity theories. In this first article we derive the field equations for an enhanced gauge Lagrangian and look for solutions of these equations. Thereby we extend earlier joint work with Tresguerres [6] and lay the foundation for the second article in this series, which will deal with the observational consequences of this new cosmological model.

The plan of this paper is as follows. In section 2 we derive the field equations and Noether identities for a new gauge Lagrangian on a formal level. After that we make use of computer algebra and provide the explicit form of the field equations and Noether identities for a rather general choice of the Weyl 1-form in section 3. In section 4 we restrict our considerations to a special case, which leads to a more manageable set of field equations. In section 5 we look for exact solutions of these equations. We will draw our conclusion in section 6 and present some plots for one branch of our model. In Appendix A and Appendix B we provide a short introduction into MAG and the Weyl-Cartan spacetime. Additionally, we show in Appendix B how our model fits into the general framework of MAG as proposed in [5]. In Appendix D, we provide an overview over the units used throughout the preceding sections. Note that we make extensive use of differential forms within this paper. A short compilation of symbols used within the paper can be found in Appendix C, for a more rigorous treatment the reader should consult Appendix A of [5].

2. Lagrangian, gauge, and matter currents

In [6] we considered the following gauge Lagrangian

$$V_{\text{old}} = \frac{\chi}{2\kappa} R_{\alpha}{}^{\beta} \wedge \eta_{\beta}{}^{\alpha} + \sum_{I=1}^6 a_I {}^{(I)}W_{\alpha}{}^{\beta} \wedge {}^*R_{\beta}{}^{\alpha} + b Z_{\alpha\beta} \wedge {}^*R^{\beta\alpha}. \quad (1)$$

Where $R_{\alpha\beta} = W_{\alpha\beta} + Z_{\alpha\beta} =$ antisymmetric + symmetric part of the curvature, and $\eta_{\alpha\beta} := {}^*(\vartheta_{\alpha} \wedge \vartheta_{\beta})$. Numbers in parentheses in front of quantities correspond to the irreducible decompositions performed in [8].

Since we are interested in more general gauge Lagrangians (cf equation (A.11) for a very general one proposed in MAG), we are going to extend (1). We will perform our calculations on the basis of a Weyl-Cartan spacetime, i.e. a spacetime in which the tracefree part of the nonmetricity $Q_{\alpha\beta}$ vanishes, a short introduction into Weyl-Cartan spacetime is given in Appendix B. The most obvious extension of (1) is given by

$$V_1 = \sum_{I=1}^4 c_I {}^{(I)}Q_{\alpha\beta} \wedge {}^*Q^{\beta\alpha}. \quad (2)$$

Since in a Weyl-Cartan space the nonmetricity is reduced to its trace part, i.e. $Q_{\alpha\beta} = g_{\alpha\beta}Q = \frac{1}{4}g_{\alpha\beta}Q^{\gamma}{}_{\gamma} = {}^{(4)}Q_{\alpha\beta}$, see equation (B.1), equation (2) now reads

$$V_1 = c_4 {}^{(4)}Q_{\alpha\beta} \wedge {}^*{}^{(4)}Q^{\beta\alpha} =: c Q_{\alpha\beta} \wedge {}^*Q^{\beta\alpha}. \quad (3)$$

Our new Lagrangian now reads

$$V = V_{\text{old}} + V_1 \quad (4)$$

$$\begin{aligned} &= \text{Einstein-Hilbert} + \text{quadratic rotational curvature} \\ &+ \text{quadratic strain curvature} + \text{quadratic nonmetricity}. \end{aligned} \quad (5)$$

In contrast to [14] we included an explicit nonmetricity term in our Lagrangian. Note that we have the arbitrary constants χ , $a_{I=1\dots 6}$, b , c , and the *weak* gravity coupling constant κ . The Lagrangian in (5) can be viewed as another step towards a better understanding of the full MAG Lagrangian as displayed in (A.11). Since a treatment of the full Lagrangian is computationally not feasible at the moment it is necessary to successively study the impact of additional terms in the Lagrangian (a review of Lagrangians used in MAG and exact solutions of the corresponding field equations can be found in [11]). Together with the quadratic rotational curvature and quadratic strain curvature terms, which were already included in our previous work [6], we now have an additional post-Riemannian piece in form of a quadratic nonmetricity term which enhances the usual Einstein-Hilbert Lagrangian commonly used in general relativistic cosmological models. Note that our Lagrangian in (5) does not include a term with the usual cosmological constant. As we will show in section 4 our ansatz in (5) gives rise to an additional constant which, on the level of the field equations, will play the same role as the cosmological constant in the standard model. Hence we omit an explicit cosmological constant term at this stage. From (5) we can derive the gauge field excitations. They read

$$M^{\alpha\beta} = -4c {}^\star Q^{\beta\alpha} = -c {}^\star (g^{\beta\alpha} Q^\gamma{}_\gamma), \quad (6)$$

$$H_\alpha = 0, \quad (7)$$

$$H^\alpha{}_\beta = -\frac{\chi}{2\kappa} \eta_\beta{}^\alpha - 2 \sum_{I=1}^6 a_I {}^\star (I) W_\beta{}^\alpha - \frac{b}{2} \delta_\beta^\alpha {}^\star R^\gamma{}_\gamma. \quad (8)$$

The canonical gauge energy-momentum is given by

$$\begin{aligned} E_\alpha &= e_\alpha \rfloor V + (e_\alpha \rfloor R_\beta{}^\gamma) \wedge H^\beta{}_\gamma + (e_\alpha \rfloor T^\beta) \wedge H_\beta + \frac{1}{2} (e_\alpha \rfloor Q_{\beta\gamma}) \wedge M^{\beta\gamma} \\ &= e_\alpha \rfloor V + (e_\alpha \rfloor R_\beta{}^\gamma) \wedge H^\beta{}_\gamma + \frac{1}{2} (e_\alpha \rfloor Q_{\beta\gamma}) \wedge M^{\beta\gamma}. \end{aligned} \quad (9)$$

In contrast to [6], we now have a non-vanishing gauge hypermomentum \ddagger

$$\begin{aligned} E^\alpha{}_\beta &= -\vartheta^\alpha \wedge H_\beta - g_{\beta\gamma} M^{\alpha\gamma} \\ &\stackrel{N}{=} 4c g_{\beta\gamma} {}^\star Q^{\gamma\alpha} = c g_{\beta\gamma} {}^\star (g^{\gamma\alpha} Q^\nu{}_\nu). \end{aligned} \quad (10)$$

The field equations now turn into

$$-E_\alpha = \Sigma_\alpha, \quad (11)$$

$$dH^\alpha{}_\alpha - E^\alpha{}_\alpha \stackrel{N}{=} \Delta, \quad (12)$$

$$g_{\gamma[\alpha} D H^\gamma{}_{\beta]} - E_{[\alpha\beta]} \stackrel{N}{=} \tau_{\alpha\beta}. \quad (13)$$

\ddagger Additional assumptions are marked with an "A". Note that we mark new relations, new with respect to [6], with an "N".

Note that in eqs. (12) and (13) we decomposed the second field equation into its trace and antisymmetric part (cf Appendix B). Since we are interested in the behaviour induced by the new part in (5), we will confine ourselves to a *non-massive medium without spin*, i.e. $\tau_{\alpha\beta} \stackrel{A}{=} 0$. Thus, eq. (13) turns into

$$g_{\gamma[\alpha} D H^{\gamma}_{\beta]} - E_{[\alpha\beta]} \stackrel{N}{=} 0. \quad (14)$$

Because we have not specified a matter Lagrangian, we have to take into account the Noether identities (cf Appendix A and Appendix B), i.e.

$$\begin{aligned} D\Sigma_{\alpha} &= (e_{\alpha} \rfloor T^{\beta}) \wedge \Sigma_{\beta} - \frac{1}{2} (e_{\alpha} \rfloor Q) \sigma^{\beta}_{\beta} + (e_{\alpha} \rfloor R_{[\beta\gamma]}) \wedge \tau^{\beta\gamma} \\ &\quad + \frac{1}{4} (e_{\alpha} \rfloor R) \wedge \Delta, \end{aligned} \quad (15)$$

$$\sigma_{\alpha\beta} = \frac{1}{4} g_{\alpha\beta} d\Delta + \vartheta_{(\alpha} \wedge \Sigma_{\beta)}, \quad (16)$$

$$0 = \vartheta_{[\alpha} \wedge \Sigma_{\beta]}. \quad (17)$$

We can rewrite eq. (16) by using (12)

$$\sigma_{\alpha\beta} \stackrel{N}{=} -\frac{1}{4} g_{\alpha\beta} dE^{\gamma}_{\gamma} + \vartheta_{(\alpha} \wedge \Sigma_{\beta)}. \quad (18)$$

Note that (16)-(17) represent the decomposed second Noether identity in case of a vanishing spin current. With (17), eq. (15) turns into

$$\begin{aligned} D\Sigma_{\alpha} &\stackrel{N}{=} (e_{\alpha} \rfloor T^{\beta}) \wedge \Sigma_{\beta} - \frac{1}{2} (e_{\alpha} \rfloor Q) \vartheta^{\beta} \wedge \Sigma_{\beta} + \frac{1}{8} (e_{\alpha} \rfloor Q) g^{\beta}_{\beta} dE^{\gamma}_{\gamma} \\ &\quad + \frac{1}{4} (e_{\alpha} \rfloor R) \wedge \Delta. \end{aligned} \quad (19)$$

Thus, we have to solve (11), (12), (14), and (17)–(19) in order to obtain a solution for our model proposed in (5). Now let us investigate the matter sources of our model. For a vanishing spin current, the hypermomentum $\Delta_{\alpha\beta}$ becomes proportional to its trace part, i.e. the dilation current of (B.4)

$$\Delta_{\alpha\beta} = \frac{1}{4} g_{\alpha\beta} \Delta^{\gamma}_{\gamma}. \quad (20)$$

The trace part of the second field equation (12) yields

$$\Delta_{\alpha\beta} = \frac{1}{4} g_{\alpha\beta} (dH^{\gamma}_{\gamma} - E^{\gamma}_{\gamma}). \quad (21)$$

In contrast to [6], the dilation current is no longer a conserved quantity since $d\Delta_{\alpha\beta} \neq 0$. From (8), and (10) we can infer that

$$\Delta_{\alpha\beta} = -\frac{1}{4} g_{\alpha\beta} (2b d^{\star} R^{\gamma}_{\gamma} + c g_{\gamma\mu}^{\star} (g^{\mu\gamma} Q^{\nu}_{\nu})). \quad (22)$$

Because of (B.10), i.e. $R^{\gamma}_{\gamma} \sim dQ$, this equation turns into

$$\Delta_{\alpha\beta} \stackrel{N}{=} -\frac{1}{4} g_{\alpha\beta} (b d^{\star} dQ^{\gamma}_{\gamma} + c g_{\gamma\mu}^{\star} (g^{\mu\gamma} Q^{\nu}_{\nu})). \quad (23)$$

Thus, for our ansatz the hypermomentum $\Delta_{\alpha\beta}$ depends on the nonmetricity and vice versa. Note that the second term in (23) depends on the coupling constant introduced

in eq. (6). Now let us specify the remaining quantities in our model. Equation (17) forces the components of energy-momentum 3-form to be symmetric, thus we choose[§]

$$\Sigma_\alpha \stackrel{A}{=} \Sigma_{\alpha\beta} \eta^\beta \quad , \quad \text{with} \quad \Sigma_{\alpha\beta} = \text{diag}(\mu(t), p_r(t), p_t(t), p_t(t)) . \quad (24)$$

Subsequently, we can calculate the metric stress-energy $\sigma_{\alpha\beta}$ from eq. (18)

$$\sigma_{\alpha\beta} = -\frac{1}{4} g_{\alpha\beta} d(c g_{\gamma\mu} {}^\star(g^{\mu\gamma} Q^\nu{}_\nu)) + \vartheta_{(\alpha} \wedge (\Sigma_{\beta)\gamma} \eta^\gamma) . \quad (25)$$

Again we obtained a quantity which depends on the Weyl 1-form, i.e. the trace of the nonmetricity. Since we want to compare our model with the cosmological standard model, we take the Robertson-Walker line element as starting point of our considerations

$$\vartheta^{\hat{0}} = dt, \quad \vartheta^{\hat{1}} = \frac{S(t)}{\sqrt{1 - kr^2}} dr, \quad \vartheta^{\hat{2}} = S(t) r d\theta, \quad \vartheta^{\hat{3}} = S(t) r \sin \theta d\phi, \quad (26)$$

with

$$ds^2 \stackrel{A}{=} \vartheta^{\hat{0}} \otimes \vartheta^{\hat{0}} - \vartheta^{\hat{1}} \otimes \vartheta^{\hat{1}} - \vartheta^{\hat{2}} \otimes \vartheta^{\hat{2}} - \vartheta^{\hat{3}} \otimes \vartheta^{\hat{3}} . \quad (27)$$

As usual, $S(t)$ denotes the cosmic scale factor and $k = -1, 0$, or 1 determines whether the three-dimensional spatial sections of spacetime are of constant negative, vanishing, or positive Riemannian curvature. Following the model proposed in [6], we will choose the torsion to be proportional to its vector piece $T^\alpha \sim {}^{(2)}T^\alpha$ and relate it to the Weyl 1-form as follows

$$T^\alpha \stackrel{A}{=} \frac{1}{2} Q \wedge \vartheta^\alpha . \quad (28)$$

The only thing missing for setting up the field equations is a proper ansatz for the Weyl 1-form Q . In [6] we were able to derive Q from an ansatz for the potential of the hypermomentum Δ , the so called polarization 2-form P . Here we will adopt a slightly different point of view. Since we are interested in the impact of different choices of the non-Riemannian quantity Q on cosmology, we will directly prescribe it in the following. Besides of the fact that we gain direct control of the post-Riemannian features of our model, we circumvent the question which type of matter might generate the corresponding hypermomentum. This question and the investigation of models with a more sophisticated matter model, like the hyperfluid of Obukhov *et al* [13], will be postponed to later articles. Let us note that our ansatz in equation (24) is in general not compatible with the energy-momentum obtained in ([13], eq. (3.28)). Both quantities are only equal in special cases like the one we will investigate in section 4. Since we do not prescribe a matter Lagrangian and use the Noether identities as constraints on the matter variables, our approach could be termed *phenomenological* as suggested in the first part of [13].

[§] Here we made use of $\eta^\alpha := {}^\star\vartheta^\alpha$.

3. Field equations and Noether identities

In this section we will derive the field equations and Noether identities resulting from specific choices of the 1-form Q which controls nearly every feature of our model. We start with a rather general form of Q , namely

$$Q = \frac{\xi(t, r)}{S(t)} \vartheta^{\hat{0}}, \quad (29)$$

where $\xi(t, r)$ denotes an arbitrary function^{||} of the radial and the time coordinate, and $S(t)$ represents the cosmic scale factor of (26). With the help of computer algebra we find that the field equations (11), (12), and (14) yield a set of four equations. In order to compare these new field equations with the ones derived in [6] (cf eqs. (40)-(43) therein) we write them as follows:

$$\begin{aligned} \chi \left(\left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right) - (a_4 + a_6) \kappa \left(\left(\frac{\ddot{S}}{S} \right)^2 - \left[\left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right]^2 \right) \\ = \frac{\kappa}{3} \left(\mu - 4c \left(\frac{\xi}{S} \right)^2 + b(1 - kr^2) \frac{\xi_{,r}^2}{S^4} \right), \end{aligned} \quad (30)$$

$$\begin{aligned} \chi \left(2 \frac{\ddot{S}}{S} + \left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right) + (a_4 + a_6) \kappa \left(\left(\frac{\ddot{S}}{S} \right)^2 - \left[\left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right]^2 \right) \\ = -\kappa \left(p_r - 4c \left(\frac{\xi}{S} \right)^2 - b(1 - kr^2) \frac{\xi_{,r}^2}{S^4} \right), \end{aligned} \quad (31)$$

$$\begin{aligned} \chi \left(2 \frac{\ddot{S}}{S} + \left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right) + (a_4 + a_6) \kappa \left(\left(\frac{\ddot{S}}{S} \right)^2 - \left[\left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right]^2 \right) \\ = -\kappa \left(p_t - 4c \left(\frac{\xi}{S} \right)^2 + b(1 - kr^2) \frac{\xi_{,r}^2}{S^4} \right), \end{aligned} \quad (32)$$

$$\frac{d}{dt} \left(\frac{\ddot{S}}{S} + \left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right) = 0. \quad (33)$$

Since we have not specified a matter Lagrangian, we have to be careful with the Noether identities (15)-(17). Our ansatz (29) yields two equations

$$\dot{S}S^3(3\mu + p_r + 2p_t) + \dot{\mu}S^4 - 16\dot{S}cS\xi^2 + 2\xi_{,rt}\xi_{,r}b(1 - kr^2) - 8\xi_{,t}cS^2\xi = 0, \quad (34)$$

$$\xi_{,rr}\xi_{,r}br(1 - kr^2) + \xi_{,r}^2b(2 - 3kr^2) + 4\xi_{,r}crS^2\xi + S^4(p_t - p_r) = 0. \quad (35)$$

Note that in eq. (33) we assumed that $a_4 \neq -a_6$. As one realizes immediately, eqs. (30)-(33) are very similar to the ones we obtained in ([6], (40)-(43)). There is only a change on the rhs, i.e. the matter side, of the above equations in form of additional

^{||} Note that this function is *not* identical with the one used in ([6], eq. (24)). It has a slightly different meaning since we use it here directly in our ansatz for the nonmetricity.

terms contributing to the pressure and energy density. As we can see from eqs. (30)-(32), the terms proportional to $\xi_{,r}$ vanish if we make the same assumptions as in [6], i.e. $\xi(t, r) \rightarrow \xi(t)$. Apart from this feature, there is another, more subtle change in (30)-(33), i.e. a term of the order ξ^2 controlled by the new coupling constant c (cf eq. (3))[¶]. The Noether identities (34) and (35) can be transformed to

$$\frac{\partial}{\partial t} (\mu S^4 + \xi_{,r}^2 b (1 - kr^2) - 8c (S\xi)^2) + 4cS^2 \frac{\partial \xi^2}{\partial t} = \frac{1}{4} \frac{dS^4}{dt} (\mu - p_r - 2p_t), \quad (36)$$

$$p_r - p_t = \frac{2cr}{S^2} \frac{\partial \xi^2}{\partial r} + \frac{b}{S^4} \left(\frac{r}{2} (1 - kr^2) \frac{\partial \xi_{,r}^2}{\partial r} + (2 - 3kr^2) \xi_{,r}^2 \right). \quad (37)$$

Comparison of (37) with ([6], eq. (37)) yields a more sophisticated relation between the radial and tangential stresses.

Let us now extract some more information from the field equations. Addition of (30) and (32) yields

$$2\chi \left(\frac{\ddot{S}}{S} + \left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right) = \frac{\kappa}{3} \left(\mu - 3p_t + 8c \left(\frac{\xi}{S} \right)^2 - 2b (1 - kr^2) \frac{\xi_{,r}^2}{S^4} \right). \quad (38)$$

Subtracting (32) from (30) yields

$$\begin{aligned} 2\chi \frac{\ddot{S}}{S} + 2\kappa (a_4 + a_6) \left(\left(\frac{\ddot{S}}{S} \right)^2 - \left[\left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right]^2 \right) \\ = -\frac{\kappa}{3} \left(\mu + 3p_t - 16c \left(\frac{\xi}{S} \right)^2 + 4b (1 - kr^2) \frac{\xi_{,r}^2}{S^4} \right). \end{aligned} \quad (39)$$

Let us now combine (33) and (38)

$$\begin{aligned} 0 &\stackrel{(33)}{=} 2\chi \frac{d}{dt} \left(\frac{\ddot{S}}{S} + \left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right) \\ &\stackrel{(38)}{=} \frac{\kappa}{3} \frac{d}{dt} \left(\mu - 3p_t + 8c \left(\frac{\xi}{S} \right)^2 - 2b (1 - kr^2) \frac{\xi_{,r}^2}{S^4} \right). \end{aligned} \quad (40)$$

The trace of the energy-momentum reads

$$\begin{aligned} \Sigma^\gamma_\gamma &= \mu - p_r - 2p_t \\ &\stackrel{(37)}{=} \mu - 3p_t - \frac{2cr}{S^2} \frac{\partial \xi^2}{\partial r} + \frac{b}{S^4} \left(\frac{r}{2} (1 - kr^2) \frac{\partial \xi_{,r}^2}{\partial r} + (2 - 3kr^2) \xi_{,r}^2 \right). \end{aligned} \quad (41)$$

Since we encountered a system of coupled PDEs, we will confine us to a special case in the following in which the field equations turn into a set of coupled ODEs. At this point we would like to note that the above situation is reminiscent to the extensions of

[¶] Formerly the term of the order ξ^2 was controlled by the coupling constant b (cf eq. (1)), and ([6], (41)-(43)).

the classical FLRW model to anisotropic and inhomogeneous metrical structures. For completeness we list the surviving curvature pieces for the ansatz in equation (29)

$${}^{(4)}W^{\alpha\beta} = \frac{\ddot{S}S - \dot{S}^2 - k}{2S^2} \vartheta^\alpha \wedge \vartheta^\beta, \quad (42)$$

$${}^{(6)}W^{\alpha\beta} = \frac{\ddot{S}S + \dot{S}^2 + k}{2S^2} \vartheta^\alpha \wedge \vartheta^\beta, \quad (43)$$

$${}^{(4)}Z_{\hat{0}\hat{0}} = -{}^{(4)}Z_{\hat{1}\hat{1}} = -{}^{(4)}Z_{\hat{2}\hat{2}} = -{}^{(4)}Z_{\hat{3}\hat{3}} = -\frac{\xi_{,r}\sqrt{-kr^2+1}}{2S^2} \vartheta^{\hat{0}} \wedge \vartheta^{\hat{1}}. \quad (44)$$

4. Special case $\xi(t, r) \rightarrow \zeta(t)$

In this section we will investigate the interesting special case in which Q , cf eq. (29), is given by a closed 1-form, i.e.

$$Q = \frac{\zeta(t)}{S(t)} \vartheta^{\hat{0}}. \quad (45)$$

The field equations are now given by

$$\begin{aligned} \chi \left(\left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right) - (a_4 + a_6) \kappa \left(\left(\frac{\ddot{S}}{S} \right)^2 - \left[\left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right]^2 \right) \\ = \frac{\kappa}{3} \left(\mu - 4c \left(\frac{\zeta}{S} \right)^2 \right), \end{aligned} \quad (46)$$

$$\begin{aligned} \chi \left(2\frac{\ddot{S}}{S} + \left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right) + (a_4 + a_6) \kappa \left(\left(\frac{\ddot{S}}{S} \right)^2 - \left[\left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right]^2 \right) \\ = -\kappa \left(p_r - 4c \left(\frac{\zeta}{S} \right)^2 \right), \end{aligned} \quad (47)$$

$$\begin{aligned} \chi \left(2\frac{\ddot{S}}{S} + \left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right) + (a_4 + a_6) \kappa \left(\left(\frac{\ddot{S}}{S} \right)^2 - \left[\left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right]^2 \right) \\ = -\kappa \left(p_t - 4c \left(\frac{\zeta}{S} \right)^2 \right), \end{aligned} \quad (48)$$

$$\frac{d}{dt} \left(\frac{\ddot{S}}{S} + \left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right) = 0. \quad (49)$$

Thus, the function ζ contributes to the energy density and pressure in a similar way as the function ξ in [6]. We note that there is no additional contribution from the strain curvature in eqs. (46)–(48), i.e. no term controlled by the coupling constant b of our Lagrangian (cf eq. (1)). This behaviour is explained by the fact that the strain curvature vanishes identically for closed 1-forms, like Q from eq. (45), in a Weyl-Cartan

spacetime. The Noether identities now read:

$$\frac{d}{dt} (\mu S^4 - 8c (S\zeta)^2) + 4cS^2 \frac{d\zeta^2}{dt} = \frac{1}{4} \frac{dS^4}{dt} (\mu - p_r - 2p_t), \quad (50)$$

$$p_r - p_t = 0. \quad (51)$$

In contrast to (37), eq. (51) forces the radial stress to be equal to the tangential stress. Addition of (46) and (48), i.e. eq. (38), yields

$$2\chi \left(\frac{\ddot{S}}{S} + \left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right) = \frac{\kappa}{3} \left(\mu - 3p_r + 8c \left(\frac{\zeta}{S} \right)^2 \right). \quad (52)$$

Subtracting (48) from (46) (cf eq. 39) yields

$$\begin{aligned} 2\chi \frac{\ddot{S}}{S} + 2\kappa (a_4 + a_6) \left(\left(\frac{\ddot{S}}{S} \right)^2 - \left[\left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right]^2 \right) \\ = -\frac{\kappa}{3} \left(\mu + 3p_r - 16c \left(\frac{\zeta}{S} \right)^2 \right). \end{aligned} \quad (53)$$

Combination of (52) with the field equations (cf eq. (40)) leads to

$$\begin{aligned} 0 &\stackrel{(49)}{=} \frac{\kappa}{3} \frac{d}{dt} \left(\mu - 3p_r + 8c \left(\frac{\zeta}{S} \right)^2 \right) \\ &= \frac{\kappa}{3} \frac{d}{dt} \left(\Sigma^\gamma_\gamma + 8c \left(\frac{\zeta}{S} \right)^2 \right) \Rightarrow \Sigma^\gamma_\gamma + 8c \left(\frac{\zeta}{S} \right)^2 = \text{const} =: \Xi \end{aligned} \quad (54)$$

Thus, we obtained a conserved quantity similar to the one in ([6], eq. (47)). The first Noether identity (50) takes the form

$$\begin{aligned} \frac{d}{dt} \left[S^4 \left(\mu - 8c \left(\frac{\zeta}{S} \right)^2 \right) \right] + 4cS^2 \frac{d\zeta^2}{dt} &= \frac{1}{4} \frac{dS^4}{dt} \Sigma^\gamma_\gamma \\ &\stackrel{(54)}{\Leftrightarrow} 4 \frac{\dot{S}}{S} \left(\Xi - \mu - \frac{3}{4} \Sigma^\gamma_\gamma \right) - \dot{\mu} = 8c \left(\frac{\zeta}{S} \right)^2 \left(2 \frac{\dot{S}}{S} - \frac{\dot{\zeta}}{\zeta} \right). \end{aligned} \quad (55)$$

Before we proceed with the search for explicit solutions, we will collect the remaining field equations

$$\begin{aligned} \chi \left(\left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right) - (a_4 + a_6) \kappa \left(\left(\frac{\ddot{S}}{S} \right)^2 - \left[\left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right]^2 \right) \\ = \frac{\kappa}{3} \left(\mu - 4c \left(\frac{\zeta}{S} \right)^2 \right), \\ \chi \left(\Lambda + \frac{\ddot{S}}{S} \right) + (a_4 + a_6) \kappa \left(\left(\frac{\ddot{S}}{S} \right)^2 - \left[\left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right]^2 \right) \end{aligned} \quad (56)$$

Table 1. Assumptions made up to this point.

Ansatz/Assumption	Resulting quantity/equation	Equation
$\tau_{\alpha\beta} = 0$	Affects the form of the second field equation	(14)
$T^\alpha = \frac{1}{2}Q \wedge \vartheta^\alpha$	Affects the form of the connection	(B.3)
$Q = \frac{\xi(t,r)}{S(t)} \vartheta^{\hat{0}}$	Controls non-Riemannian features/ Affects the form of the field equations	(23),(29),(B.1), (30)-(35)
$Q = \frac{\zeta(t)}{S(t)} \vartheta^{\hat{0}}$	Controls non-Riemannian features/ Simplifies field equations	(23),(45),(B.1), (46)-(51)
$a_4 \neq -a_6$	Affects the form of the second field equation	(33)
$\Xi = \Sigma^\alpha{}_\alpha = 0$	Relation between μ and p_r	(60)
Λ	Affects the form of the field equations	(57)-(58)

$$= -\kappa \left(p_r - 4c \left(\frac{\zeta}{S} \right)^2 \right), \quad (57)$$

$$\frac{\ddot{S}}{S} + \left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} = \Lambda, \quad (58)$$

$$4 \frac{\dot{S}}{S} \left(\Xi - \mu - \frac{3}{4} \Sigma^\gamma{}_\gamma \right) - \dot{\mu} = 8c \left(\frac{\zeta}{S} \right)^2 \left(2 \frac{\dot{S}}{S} - \frac{\dot{\zeta}}{\zeta} \right). \quad (59)$$

Note that the new constant Λ in (57) is defined via (58). Comparison of (58) with the Friedman equation in standard cosmology reveals that Λ plays the same role as the usual cosmological constant. Since we did not include this additional constant in our Lagrangian right from the beginning Λ might be termed *induced* cosmological constant. Now let us exploit the fact that we are allowed to set the constant $\Xi \stackrel{A}{=} 0$, which leads to an additional constraint, i.e.

$$\mu = 3p_r - 8c \left(\frac{\zeta}{S} \right)^2. \quad (60)$$

Subsequently eq. (56) turns into

$$\begin{aligned} & \chi \left(\left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right) - (a_4 + a_6) \kappa \left(\left(\frac{\ddot{S}}{S} \right)^2 - \left[\left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right]^2 \right) \\ & = \kappa \left(p_r - 4c \left(\frac{\zeta}{S} \right)^2 \right), \end{aligned} \quad (61)$$

and the second Noether identity (59) now reads

$$4 \frac{\dot{S}}{S} \mu + \dot{\mu} = 8c \left(\frac{\zeta}{S} \right)^2 \left(\frac{\dot{S}}{S} + \frac{\dot{\zeta}}{\zeta} \right). \quad (62)$$

Note that we collected all assumptions made up to this point in table 1.

5. Solutions

5.1. $\Lambda \neq 0$ solutions

We are now going to solve eq. (58) for nonvanishing Λ . Note that eq. (58) does not depend on the relation between the energy density and pressure and therefore can be solved independently. This ODE, after a substitution, turns into a Bernoulli ODE, which in turn can be transformed into a linear equation. After this procedure we obtain two branches for the scale factor. They read as follows:

$$S = \pm \frac{1}{\sqrt{2\Lambda}} \sqrt{e^{-\sqrt{2\Lambda}t} \left(2ke^{\sqrt{2\Lambda}t} - \sqrt{2\Lambda}\varkappa_1 e^{2\sqrt{2\Lambda}t} + \sqrt{2\Lambda}\varkappa_2 \right)}, \quad (63)$$

where \varkappa_1 , and \varkappa_2 are constants. This solution for the scale factor is valid for all three possible choices of k . Let us now proceed by fixing the equation of state.

We will start with the most simple ansatz, i.e. with the introduction of an additional constant w into the equation of state, which parameterizes the ratio of the energy density and the pressure in our model,

$$w \mu(t) = p_r(t). \quad (64)$$

Now let us derive the impact of (64) on our set of field equations given by (56)–(62). Equation (60) yields

$$\mu = -\frac{8c}{1-3w} \left(\frac{\zeta}{S} \right)^2. \quad (65)$$

The field equations are now given by

$$\begin{aligned} \chi \left(\left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right) - (a_4 + a_6) \kappa \left(\left(\frac{\ddot{S}}{S} \right)^2 - \left[\left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right]^2 \right) \\ = -\frac{4c\kappa}{3} \left(\frac{\zeta}{S} \right)^2 \left(\frac{1+3w}{1-3w} \right), \end{aligned} \quad (66)$$

$$\begin{aligned} \chi \left(\Lambda + \frac{\ddot{S}}{S} \right) + (a_4 + a_6) \kappa \left(\left(\frac{\ddot{S}}{S} \right)^2 - \left[\left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} \right]^2 \right) \\ = 4c\kappa \left(\frac{\zeta}{S} \right)^2 \left(\frac{1-w}{1-3w} \right), \end{aligned} \quad (67)$$

$$\frac{\ddot{S}}{S} + \left(\frac{\dot{S}}{S} \right)^2 + \frac{k}{S^2} = \Lambda, \quad (68)$$

$$\frac{24c(1-w)}{3w-1} \left(\frac{\zeta}{S} \right)^2 \left(\frac{\dot{S}}{S} + \frac{\dot{\zeta}}{\zeta} \right) = 0. \quad (69)$$

Equation (69) has two non-trivial solution, namely

$$\zeta = \frac{\iota}{S}, \text{ with } \iota = \text{const}, \quad \text{and} \quad w = 1. \quad (70)$$

Table 2. Ansatz $\Lambda \neq 0, w = \text{const.}$

ζ	S	Additional constraints
$\zeta = \frac{\iota}{S}$	S from eq. (63)	$\{a_4 = a_6 = c = \chi = 0\},$ $\{a_4 = a_6 = \iota = 0, w = \frac{1}{3}\},$ $\{a_4 = a_6 = \chi = 0, w = 1\},$ $\{a_4 = a_6 = c = 0, w = \frac{1}{3}\},$ $\{a_4 = -a_6, \Lambda = 0, w = \text{const}\}$

Table 3. Ansatz $\Lambda \neq 0, w = w(t).$

ζ	S	Additional constraints
ζ arbitrary, $w = \frac{\zeta^2 S^2 \varkappa_3 - 1}{\zeta^2 S^2 \varkappa_3 - 3}$	S from eq. (63)	$\{a_4 = -a_6, \Lambda = 0\},$ $\{a_4 = -a_6, \chi = 0, w = 1\},$ $\{a_4 = -a_6, \iota = \chi = 0\}$

Solving the remaining field equations with respect to the first solution in eq. (70), we obtain constraints among the coupling constants which are summarized in table 2 (note that every set of parameters on the rhs corresponds to a solution of the field equations). These solutions are not very satisfactory since they either lead to vanishing post-Riemannian quantities or, in case of $\chi = c = 0$, to a restriction on the Lagrangian level.

Let us switch to another ansatz for the equation of state, namely

$$w(t)\mu(t) = p_r(t). \quad (71)$$

Thus, we introduced an additional function into the equation of state which controls the relation between the energy density and stresses in a *dynamical* way. The field equation which changes with respect to the set (66)–(69), besides of the fact that w is no longer a constant, is the Noether identity in eq. (69), which now reads

$$-\frac{24c}{(3w-1)^2} \left(\frac{\zeta}{S}\right)^2 \left(\frac{\dot{S}}{S} (1+3w^2-4w) + \frac{\dot{\zeta}}{\zeta} (1+3w^2-4w) + \dot{w}\right) = 0. \quad (72)$$

In case of an arbitrary choice of ζ , this equation is solved by

$$w = \frac{S^2 \zeta^2 \varkappa_3 - 1}{S^2 \zeta^2 \varkappa_3 - 3}. \quad (73)$$

Reinsertion of this solution for w into the remaining field equations yields additional parameter constraints which are summarized in table 3. As one realizes immediately, none of the solutions collected in table 3 is of use for us, since they all lead to unrealistic or forbidden restrictions among the coupling constants in our model. Therefore, in the following section, we will switch to the case in which the induced cosmological constant Λ vanishes.

Table 4. Ansatz $\Lambda = 0$, $w = w(t)$.

ζ	S	Additional constraints
$\zeta = \frac{t}{S}$, $w = \text{const}$	$k \neq 0$, S from eq. (74)	$w = \frac{4cl^2\kappa + \varkappa_1\chi}{4cl^2\kappa + 3\varkappa_1\chi}$
	$k = 0$, $S = \text{const}$ cf eq. (75)	$w = 1$
	$k = 0$, S from eq. (75)	$w = \frac{4cl^2\kappa + \varkappa_1^2\chi}{4cl^2\kappa + 3\varkappa_1^2\chi}$
$w = \frac{\zeta^2 S^2 \varkappa_3 - 1}{\zeta^2 S^2 \varkappa_3 - 3}$, ζ arbitrary	$k \neq 0$, S from eq. (74)	$\varkappa_3 = -4 \frac{c\kappa}{\varkappa_1\chi}$
	$k = 0$, $S = \text{const}$ cf eq. (75)	$c = 0$
	$k = 0$, S from eq. (75)	$\varkappa_3 = -4 \frac{c\kappa}{\varkappa_1^2\chi}$

5.2. $\Lambda = 0$ solutions

Solving equation (58) for vanishing Λ , yields a solution for the scale factor which depends on the value of the constant k :

$$k \neq 0 : \quad S = \pm \sqrt{\frac{1}{k} (\varkappa_1 - k^2 (\varkappa_2 + t)^2)}, \quad \text{with} \quad \varkappa_1, \varkappa_2 = \text{const}, \quad (74)$$

$$k = 0 : \quad S = \varkappa_1 \text{ or } S = \pm \sqrt{2\varkappa_1 (t + \varkappa_2)}, \quad \text{with} \quad \varkappa_1, \varkappa_2 = \text{const}. \quad (75)$$

Motivated by the results in the previous section for the $\Lambda \neq 0$ case, we will directly start with the more general equation of state as given in (71). The field equations are now given by eqs. (66)–(69) but with $\Lambda = 0$. The parameter constraints for this solution are summarized in the second part of table 4.

Additionally, we investigated the case in which made use of the old solution for ζ , i.e. as given in (70). With this ansatz for ζ the second Noether identity, as given in eq. (72), turns into:

$$\dot{w} = 0. \quad (76)$$

Thus, w has to be a constant which, subsequently, can be determined from the remaining field equations after choosing the branch for S from eqs. (74)–(75). The additional constraints for this parameter choice are listed in the first part of table 4. Most interestingly it turns out that the parameter w , which controls the equation of state, is restricted by the choice of a certain set of constants in our theory (cf rhs in table 4).

6. Conclusion

In the the last section we have shown that it is possible to find exact solutions of the field equations within our model. We were able to generate a rather broad class of solutions which allows for a flexible equation of state. We collected the resulting constraints of the parameters in our model in tables 2–4. There seem to be *no* reasonable solutions in case of a non-vanishing induced cosmological constant Λ , unless one wants to introduce strong restrictions on the Lagrangian level. Thus, we are going to focus on the solutions with *vanishing* Λ in the following.

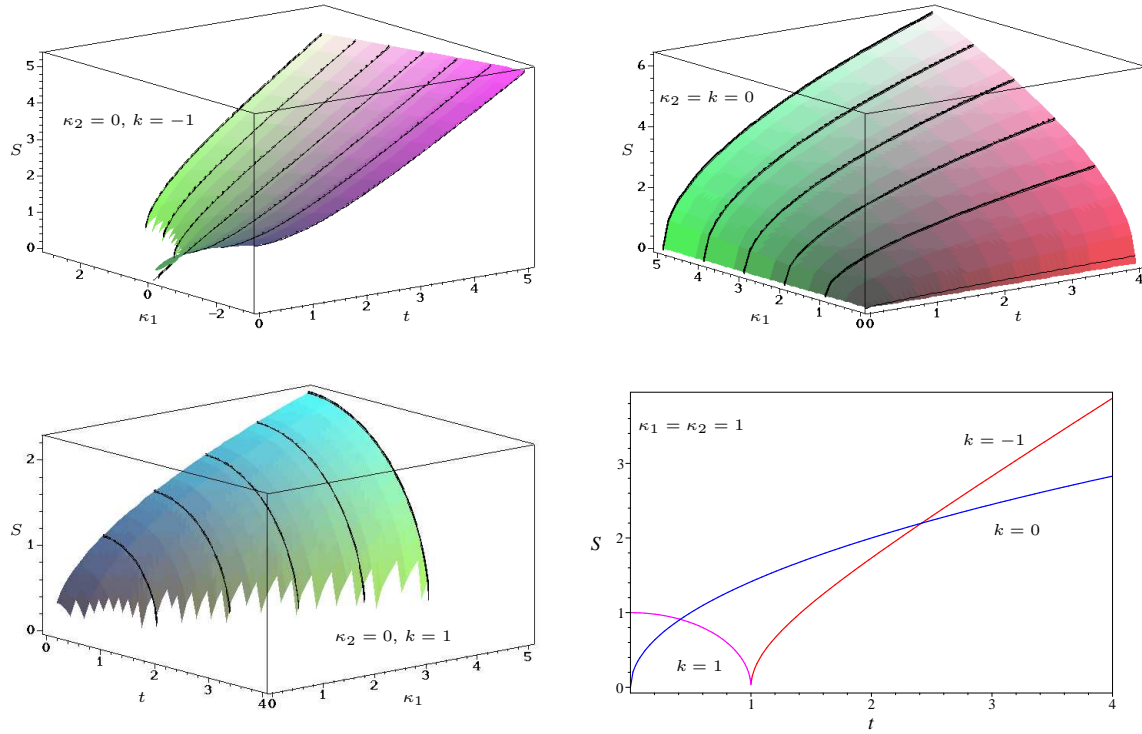


Figure 1. Temporal behaviour of the scale factor in the case of the $\Lambda = 0$ branch of the model (we always select the positive sign in front of the scale factor, cf eqs. (74)–(75)).

In figure 1 we plotted the scale factor for all three possible values of k and for different values of the parameter κ_1 . As becomes clear from the plot at bottom right, we have three qualitatively different behaviours depending on the value of k . As in the Friedman case the collapsing scenario corresponds to a universe with positive spatial curvature. In figure 2 we plotted the function Q for the ansatz mentioned in equation (70). As stated before Q , the Weyl 1-form controls the non-Riemannian features of our model. From the plots it becomes clear that it is possible to construct models in which Q vanishes at later times. Thus, the non-Riemannian quantities *die out* with time. This is a rather desirable behaviour, since the spacetime we are living in nowadays seems to be a Riemannian one. At least all experiments carried out so far point into this direction [3]. Nevertheless our model is flexible enough to cope with both situations, i.e. if there is evidence for non-Riemannian structures at the present time, we are able to implement this fact by modifying our ansatz in (29) and (45), respectively.

In comparison with the usual FLRW model of cosmology we still have three distinct cases for the evolution of the scale factor, which correspond to the three different choices for k in the ansatz for the metric in equation (26). Since one of our field equations (58) is very similar to the Friedman equation in standard cosmology we obtain a similar root type behaviour for the scale factor as displayed in figure 1. As shown in (63) an induced cosmological constant leads to inflationary like solutions. In contrast to our old model [6]

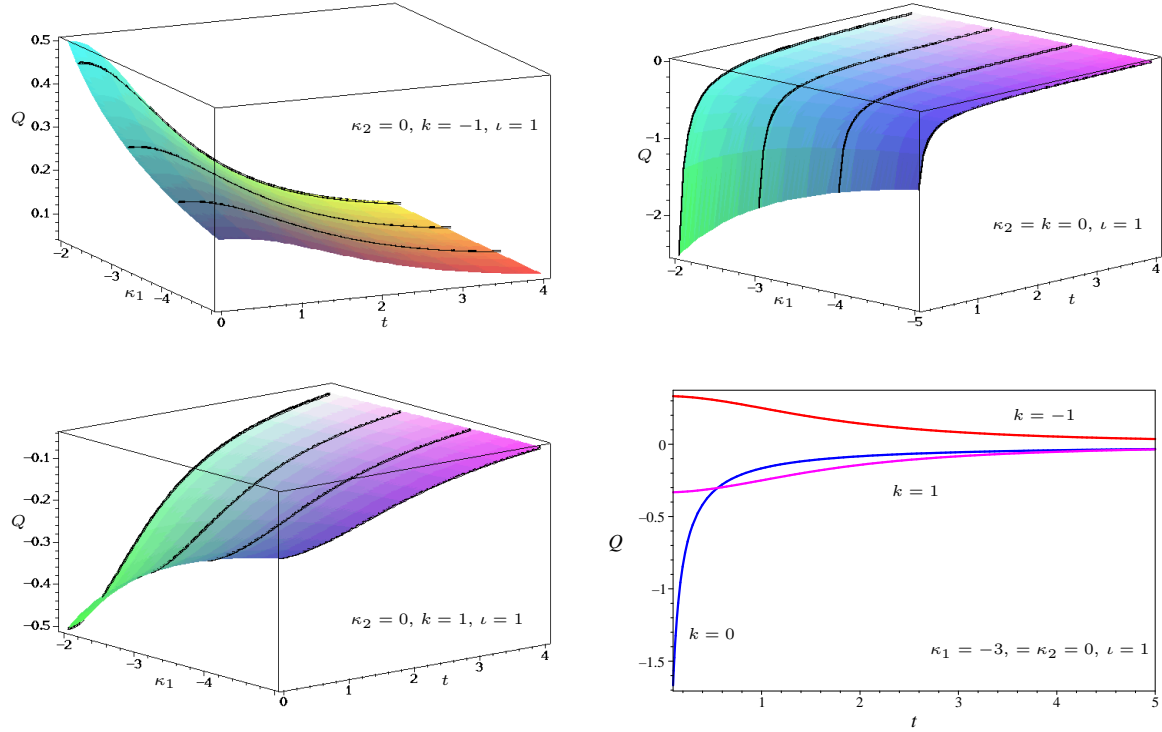


Figure 2. Temporal behaviour of the non-Riemannian Weyl 1-form Q in the case of the $\Lambda = 0, \zeta = \frac{\ell}{S}$ branch of the model (we always select the positive sign in front of the scale factor, cf eqs. (74)–(75)).

we were not able to find meaningful parameter constraints for this branch of the model (cf. tables 2 and 3). This drawback might be relaxed in the future if we switch to another ansatz for the Weyl 1-form Q . Most interestingly the non-Riemannian quantities lead to a contribution to the total energy density of the universe as shown in (65). Thus, the energy density μ is no longer a quantity which is determined by the evolution of the scale factor only, like in the FLRW scenario. As we will show in the next article of this series this contribution might be used to define a new energy density parameter which adds to the total energy budget of the universe. Thereby leading to an interesting new source for a possible dark energy component. Since the field equations differ from the Friedman equations one can expect several observational changes with respect to the standard FLRW model. Note that an ansatz with a position dependent Weyl 1-form at very early stages of the universe might contribute to the observed inhomogeneities in the cosmic microwave background. Although speculative at this time, small inhomogeneities in the new geometric quantities might also have served as seeds for structure formation at early stages, thereby yielding an interesting supplement to the quantities within the standard paradigm.

Let us summarize that the solutions found above contribute to the collection of known exact solutions in MAG, see [11]. Additionally, we managed to extend the model proposed in [6]. We provided the foundation for upcoming articles which will deal with

the observational consequences of this new model. The most pressing task will be to look for realistic parameter choices in order to determine whether the model is in agreement with recent observational data. Within the next article of this series [7], we will use the supernova data of Perlmutter *et al* [15] and Schmidt *et al* [16] in order to constrain the free parameters in our model.

Appendix A. MAG in general

In MAG we have the metric $g_{\alpha\beta}$, the coframe ϑ^α , and the connection 1-form Γ_α^β (with values in the Lie algebra of the four-dimensional linear group $GL(4, R)$) as new independent field variables. Here $\alpha, \beta, \dots = 0, 1, 2, 3$ denote (anholonomic) frame indices. Spacetime is described by a metric-affine geometry with the gravitational field strengths nonmetricity $Q_{\alpha\beta} := -Dg_{\alpha\beta}$, torsion $T^\alpha := D\vartheta^\alpha$, and curvature $R_\alpha^\beta := d\Gamma_\alpha^\beta - \Gamma_\alpha^\gamma \wedge \Gamma_\gamma^\beta$. A Lagrangian formalism for a matter field Ψ minimally coupled to the gravitational potentials $g_{\alpha\beta}, \vartheta^\alpha, \Gamma_\alpha^\beta$ has been set up in [5]. The dynamics of an ordinary MAG theory is specified by a total Lagrangian

$$L = V_{\text{MAG}}(g_{\alpha\beta}, \vartheta^\alpha, Q_{\alpha\beta}, T^\alpha, R_\alpha^\beta) + L_{\text{mat}}(g_{\alpha\beta}, \vartheta^\alpha, \Psi, D\Psi). \quad (\text{A.1})$$

The variation of the action with respect to the independent gauge potentials leads to the field equations:

$$\frac{\delta L_{\text{mat}}}{\delta \Psi} = 0, \quad (\text{A.2})$$

$$DM^{\alpha\beta} - m^{\alpha\beta} = \sigma^{\alpha\beta}, \quad (\text{A.3})$$

$$DH_\alpha - E_\alpha = \Sigma_\alpha, \quad (\text{A.4})$$

$$DH^\alpha_\beta - E^\alpha_\beta = \Delta^\alpha_\beta. \quad (\text{A.5})$$

Equations (A.3) and (A.4) are the generalized Einstein equations with the symmetric energy-momentum 4-form $\sigma^{\alpha\beta}$ and the canonical energy-momentum 3-form Σ_α as sources. Equation (A.5) is an additional field equation which takes into account other aspects of matter, such as spin, shear and dilation currents, represented by the hypermomentum Δ^α_β . We made use of the definitions of the gauge field excitations,

$$H_\alpha := -\frac{\partial V_{\text{MAG}}}{\partial T^\alpha}, \quad H^\alpha_\beta := -\frac{\partial V_{\text{MAG}}}{\partial R_\alpha^\beta}, \quad M^{\alpha\beta} := -2\frac{\partial V_{\text{MAG}}}{\partial Q_{\alpha\beta}}, \quad (\text{A.6})$$

of the canonical energy-momentum, the metric stress-energy, and the hypermomentum current of the gauge fields,

$$E_\alpha := \frac{\partial V_{\text{MAG}}}{\partial \vartheta^\alpha}, \quad m^{\alpha\beta} := 2\frac{\partial V_{\text{MAG}}}{\partial g_{\alpha\beta}}, \quad E^\alpha_\beta = -\vartheta^\alpha \wedge H_\beta - g_{\beta\gamma} M^{\alpha\gamma}, \quad (\text{A.7})$$

and of the canonical energy-momentum, the metric stress-energy, and the hypermomentum currents of the matter fields,

$$\Sigma_\alpha := \frac{\delta L_{\text{mat}}}{\delta \vartheta^\alpha}, \quad \sigma^{\alpha\beta} := 2\frac{\delta L_{\text{mat}}}{\delta g_{\alpha\beta}}, \quad \Delta^\alpha_\beta := \frac{\delta L_{\text{mat}}}{\delta \Gamma_\alpha^\beta}. \quad (\text{A.8})$$

Provided the matter equations (A.2) are fulfilled, the following Noether identities hold:

$$D\Sigma_\alpha = (e_\alpha \rfloor T^\beta) \wedge \Sigma_\beta - \frac{1}{2} (e_\alpha \rfloor Q_{\beta\gamma}) \sigma^{\beta\gamma} + (e_\alpha \rfloor R_\beta{}^\gamma) \wedge \Delta^\beta{}_\gamma, \quad (\text{A.9})$$

$$D\Delta^\alpha{}_\beta = g_{\beta\gamma} \sigma^{\alpha\gamma} - \vartheta^\alpha \wedge \Sigma_\beta. \quad (\text{A.10})$$

They show that the field equation (A.3) is redundant, thus we only need to take into account (A.4) and (A.5).

As suggested in [11], the most general parity conserving quadratic Lagrangian expressed in terms of the irreducible pieces of the nonmetricity $Q_{\alpha\beta}$, torsion T^α , and curvature $R_{\alpha\beta}$ reads

$$\begin{aligned} V_{\text{MAG}} = & \frac{1}{2\kappa} \left[-a_0 R^{\alpha\beta} \wedge \eta_{\alpha\beta} - 2\lambda \eta + T^\alpha \wedge \star \left(\sum_{I=1}^3 a_I {}^{(I)}T_\alpha \right) \right. \\ & + Q_{\alpha\beta} \wedge \star \left(\sum_{I=1}^4 b_I {}^{(I)}Q^{\alpha\beta} \right) \\ & + b_5 {}^{(3)}Q_{\alpha\gamma} \wedge \vartheta^\alpha \wedge \star ({}^{(4)}Q^{\beta\gamma} \wedge \vartheta_\beta) \\ & \left. + 2 \left(\sum_{I=2}^4 c_I {}^{(I)}Q_{\alpha\beta} \right) \wedge \vartheta^\alpha \wedge \star T^\beta \right] \\ & - \frac{1}{2\rho} R^{\alpha\beta} \wedge \star \left[\sum_{I=1}^6 w_I {}^{(I)}W_{\alpha\beta} + \sum_{I=1}^5 z_I {}^{(I)}Z_{\alpha\beta} + w_7 \vartheta_\alpha \wedge (e_\gamma \rfloor {}^{(5)}W^\gamma{}_\beta) \right. \\ & \left. + z_6 \vartheta_\gamma \wedge (e_\alpha \rfloor {}^{(2)}Z^\gamma{}_\beta) + \sum_{I=7}^9 z_I \vartheta_\alpha \wedge (e_\gamma \rfloor {}^{(I-4)}Z^\gamma{}_\beta) \right]. \quad (\text{A.11}) \end{aligned}$$

The constants entering (A.11) are the cosmological constant λ , the weak and strong coupling constant κ and ρ^+ , and the 28 dimensionless parameters

$$a_0, \dots, a_3, b_1, \dots, b_5, c_2, \dots, c_4, w_1, \dots, w_7, z_1, \dots, z_9. \quad (\text{A.12})$$

This Lagrangian and the presently known exact solutions in MAG have been reviewed in [11]. We note that this Lagrangian incorporates the one used in section 2 eq. (4), as can be seen easily by making the following choice for the constants in (A.11):

$$\lambda, a_1, \dots, a_3, b_1, \dots, b_3, b_5, c_2, \dots, c_4, w_7, z_1, \dots, z_3, z_5, \dots, z_9 = 0. \quad (\text{A.13})$$

In order to obtain exactly the form of (4), one has to perform the additional substitutions:

$$a_0 \rightarrow -\chi, w_1, \dots, w_6 \rightarrow -2\rho a_1, \dots, -2\rho a_6, b_4 \rightarrow c, z_4 \rightarrow -2\rho b. \quad (\text{A.14})$$

In table A1 we collected some of symbols defined within this appendix.

⁺ $[\lambda] = \text{length}^{-2}$, $[\kappa] = \text{length}^2$, $[\rho] = [\hbar] = [c] = 1$.

Table A1. Summary of definitions made in Appendix A.

Potentials	Field strengths	Excitations	Gauge currents
$g_{\alpha\beta}$	$Q_{\alpha\beta} := -Dg_{\alpha\beta}$	$M^{\alpha\beta} := -2\frac{\partial V}{\partial Q_{\alpha\beta}}$	$m^{\alpha\beta} := 2\frac{\partial V}{\partial g_{\alpha\beta}}$
ϑ^α	$T^\alpha := D\vartheta^\alpha$	$H_\alpha := -\frac{\partial V}{\partial T^\alpha}$	$E_\alpha := \frac{\partial V}{\partial \vartheta^\alpha}$
$\Gamma_\alpha{}^\beta$	$R_\alpha{}^\beta := D^\beta \Gamma_\alpha{}^\beta$	$H^\alpha{}_\beta := -\frac{\partial V}{\partial R_\alpha{}^\beta}$	$E^\alpha{}_\beta := \frac{\partial V}{\partial \Gamma_\alpha{}^\beta}$

Appendix B. Weyl-Cartan spacetime

The Weyl-Cartan spacetime (Y_n) is a special case of the general metric-affine geometry in which the tracefree part $\mathcal{Q}_{\alpha\beta}$ of the nonmetricity $Q_{\alpha\beta}$ vanishes. Thus, the whole nonmetricity is proportional to its trace part, i.e. the Weyl 1-form $Q := \frac{1}{4}Q^\alpha{}_\alpha$,

$$Q_{\alpha\beta} = g_{\alpha\beta} Q = \frac{1}{4}g_{\alpha\beta} Q^\gamma{}_\gamma. \quad (\text{B.1})$$

Therefore the general MAG connection reduces to

$$\begin{aligned} \Gamma_{\alpha\beta} = & \frac{1}{2}dg_{\alpha\beta} + (e_{[\alpha}dg_{\beta]\gamma})\vartheta^\gamma + e_{[\alpha}C_{\beta]} - \frac{1}{2}(e_\alpha]e_\beta]C_\gamma)\vartheta^\gamma \\ & - e_{[\alpha}T_{\beta]} + \frac{1}{2}(e_\alpha]e_\beta]T_\gamma)\vartheta^\gamma + \frac{1}{2}g_{\alpha\beta}Q + (e_{[\alpha}Q)\vartheta_{\beta]} \end{aligned} \quad (\text{B.2})$$

$$= \Gamma_{\alpha\beta}^{\{\}} - e_{[\alpha}T_{\beta]} + \frac{1}{2}(e_\alpha]e_\beta]T_\gamma)\vartheta^\gamma + \frac{1}{2}g_{\alpha\beta}Q + (e_{[\alpha}Q)\vartheta_{\beta]}. \quad (\text{B.3})$$

Thus, it does not include any more a symmetric tracefree part. Let us now recall the definition of the material hypermomentum $\Delta_{\alpha\beta}$ given in (A.8). Due to the absence of a symmetric tracefree piece in (B.3), $\Delta_{\alpha\beta}$ decomposes as follows

$$\begin{aligned} \Delta_{\alpha\beta} &= \text{antisymmetric piece} + \text{trace piece} \\ &= \tau_{\alpha\beta} + \frac{1}{4}g_{\alpha\beta}\Delta = \tau_{\alpha\beta} + \frac{1}{4}g_{\alpha\beta}\Delta^\gamma{}_\gamma \\ &= \text{spin current} + \text{dilation current}. \end{aligned} \quad (\text{B.4})$$

According to (B.4) the second MAG field equation (A.5) decomposes into

$$dH^\alpha{}_\alpha - E^\alpha{}_\alpha = \Delta, \quad (\text{B.5})$$

$$g_{\gamma[\alpha}DH^\gamma{}_{\beta]} - E_{[\alpha\beta]} = \tau_{\alpha\beta}, \quad (\text{B.6})$$

while the first field equation is still given by (A.4). Additionally, we can decompose the second Noether identity (A.10) into

$$\frac{1}{4}g_{\alpha\beta}d\Delta + \vartheta_{(\alpha}\wedge\Sigma_{\beta)} = \sigma_{\alpha\beta}, \quad (\text{B.7})$$

$$D\tau_{\alpha\beta} + Q\wedge\tau_{\alpha\beta} + \vartheta_{[\alpha}\wedge\Sigma_{\beta]} = 0. \quad (\text{B.8})$$

Thus, the first Noether identity (A.9) with inserted Weyl 1-form and hypermomentum reads

$$D\Sigma_\alpha = (e_\alpha]T^\beta)\wedge\Sigma_\beta - \frac{1}{2}(e_\alpha]Q)\sigma^\beta{}_\beta + (e_\alpha]R_{[\beta\gamma]})\wedge\tau^{\beta\gamma} + \frac{1}{4}(e_\alpha]R)\wedge\Delta. \quad (\text{B.9})$$

Table C1. Operators.

Operation	Symbol	Input	Output
Exterior multiplication	\wedge	p -form \wedge q -form	\rightarrow $(p + q)$ -form
Interior multiplication	\lrcorner	vector \lrcorner p -form	\rightarrow $(p - 1)$ -form
Exterior derivative	d	d p -form	\rightarrow $(p + 1)$ -form
Hodge star in a n -dimen. space	$*$	$*p$ -form	\rightarrow $(n - p)$ -form

Table D1. Natural units.

[energy]	[mass]	[time]	[length]
length ⁻¹	length ⁻¹	length	length

Finally, we note that in a Y_n spacetime the symmetric part of the curvature $R_{(\alpha\beta)} = Z_{\alpha\beta}$, i.e. the strain curvature, reduces to the trace part

$$Z_{\alpha\beta} = \frac{1}{4}g_{\alpha\beta}R = \frac{1}{4}g_{\alpha\beta}R^\gamma{}_\gamma = \frac{1}{2}g_{\alpha\beta}dQ. \quad (\text{B.10})$$

Appendix C. Differential geometric formalism

We assume a connected n -dimensional differential manifold Y_n as underlying structure throughout the paper. A vector basis of its tangent space $T_p Y_n$ is denoted by e_α , which is dual (i.e. $e_\alpha \lrcorner \vartheta^\beta = \delta_\alpha^\beta$) to the basis ϑ^α of the cotangent space $T_p^* Y_n$. A p -form Ξ can be expanded with respect to this basis as follows

$$\Xi = \frac{1}{p!} \Xi_{\beta_1 \dots \beta_p} \vartheta^{\beta_1} \wedge \dots \wedge \vartheta^{\beta_p}. \quad (\text{C.1})$$

Table C1 provides a rough overview of the operators used throughout the paper. For a more comprehensive treatment the reader should consult [4] or section 3, and Appendix A of [5].

Appendix D. Units

In this work we made use of *natural units*, i.e. $\hbar = c = 1$ (cf table D1). Additionally, we have to be careful with the coupling constants and the coordinates within the coframe. In order to keep things as clear as possible, we provide a list of the quantities emerging throughout all sections in table D2. Note that $[d] = 1$ and $[*] = \text{length}^{n-2p}$, where $n =$ dimension of the spacetime, $p =$ degree of the differential form on which $*$ acts.

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Table D2. Units of quantities.

Quantities	l	Length
Gauge potentials	$[g_{\alpha\beta}], [\Gamma_{\alpha}^{\beta}]$	$[\vartheta^{\alpha}]$
Gauge field strengths	$[Q_{\alpha\beta}], [R_{\alpha\beta}]$	$[T^{\alpha}]$
Gauge field excitations	$[M^{\alpha\beta}], [H^{\alpha}_{\beta}]$	$[H_{\alpha}]^{-1}$
Gauge field currents	$[E^{\alpha}_{\beta}], [m^{\alpha\beta}]$	$[E_{\alpha}]^{-1}$
Matter currents	$[\Delta_{\alpha\beta}], [\sigma^{\alpha\beta}], [\tau_{\alpha\beta}]$	$[\Sigma_{\alpha}]^{-1}$
Coordinates	$[\theta], [\phi], [r]$	$[t]$
Functions	$[\xi(t, r)], [\zeta(t)]$	$[S(t)], [\mu(t)]^{-\frac{1}{4}}, [p_r(t)]^{-\frac{1}{4}}$ $[p_t(t)]^{-\frac{1}{4}}$
Miscellany		$[\Sigma_{\alpha\beta}]^{-\frac{1}{4}}$
Constants	$[\chi], [b], [k], [a_I]$	$[\kappa]^{\frac{1}{2}}, [\Lambda]^{-\frac{1}{2}}, [c_I]^{-\frac{1}{2}}, [c]^{-\frac{1}{2}}$ $[\Xi]^{-\frac{1}{4}}$
eq. (74)		$[\varkappa_1]^{\frac{1}{2}}, [\varkappa_2]$
eq. (75)		$[\varkappa_1], [\varkappa_2]$
eq. (63)		$[\varkappa_1], [\varkappa_2]$
eq. (64)	$[w]$	
eq. (70)		$[l]$
eq. (73)		$[\varkappa_3]^{-\frac{1}{2}}$

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